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Roots of Certain Transcendental Equations for Elastic Angular Regions

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Abstract

Certain important transcendental equations occur in the case of elastic angular regions while analyzing these regions for flexure, vibration, and buckling. Previously these transcendental equations have been solved for roots and the data has been tabulated for different boundary conditions, as the angle of the region is varied. The purpose of this paper is to demonstrate that once we solve for the roots at a specific angular region, the roots for angular regions with angles ranging from 0 to 2π can be obtained via forward integration.

Key Words: Transcendental equation, elastic angular region, flexure, vibration, buckling , polygonal plate

1. Introduction

Certain transcendental equations occur in the case of solution of polygonal plate problems dealing with flexure, vibration, and buckling [1-3]. The roots of these equations need to be evaluated to form eigenfunctions which are essential for the solution of the mode shape of the plate during flexure or vibration. Also the eigenfunctions can be used to find the frequencies of vibration of sectorial, triangular, quadrilateral, and other polygonal plates.

Several different methods have been used in the past to find the frequencies of vibration of plates. An overview of the some of the studies on the free vibration of plates is given in [4]. Ref. 5 derives an equation for finding the eigenfrequencies of polygonal plates with free simply supported mixed edges and Ref. 6 analyzes the free vibration of right triangular plates using a superposition method. In [7], the Ritz method is used to find the fundamental frequencies of five-sided plates which are obtained by cutting out an isosceles triangle from one corner. The dynamical analogy with membranes is used in [8] to study the free vibration of regular polygonal plates with simply supported edges. In [9] finite elements are used to analyze annular sectorial plates having their inner circular edges clamped. All of these plates under arbitrary boundary conditions can be analyzed with the method of the eigenfunctions given in [2]. Although we emphasize the classical problems related to homogeneous isotropic thin plates under small deflections, there may be suitable extensions of the methodology to the anisotropic, nonhomogeneous and composite material cases listed in [10].

Ref. 11 tabulates the first ten complex roots of these transcendental equations as the angle of the angular region is varied from $15°$ to $180°$ in steps of $5°$. Also if real roots are present, these are also tabulated in ascending order. In this paper we show that once the roots are known at a specific angle, the roots at all the other angles ranging from 0 to 2π can be found by solving a differential equation which relates the roots to the angle of the region.

2. Transcendental Equations

In this section, we indicate how the transcendental equation for an angular region arises while solving for basis functions for the clamped case. Once the basis functions are formed, the solution for flexure or vibration of the angular region can be expressed as a linear combination of the basis functions. Since any polygonal plate can be subdivided into angular regions, the solution on the polygonal plate can be obtained by requiring the solutions on the angular regions to satisfy continuity conditions along suitable diagonals.

Let the vertex of the angular region be at $(0,0)$ with one of its edges coinciding with the x-axis. Let α be the angle of the angular region. The equation of motion of the angular plate is given by [12]

$$\nabla^4\psi + \frac{\rho}{D}\frac{\partial^2\psi}{\partial t^2} = 0$$

where D is the flexural rigidity, ρ is the uniform mass density per unit area, ψ is the transverse displacement, and $\nabla^4 = \nabla^2\nabla^2$, ∇^2 being the Laplacian.

Requiring simple harmonic vibrations $\psi = w(r,\theta)e^{-i\omega t}$, we get

$$\nabla^4 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 w = \mu w$$

where $\mu = \dfrac{\rho\omega^2}{D}$. The edges of the angular region can be assigned any of the classical homogeneous boundary conditions. As an example, the clamped boundary conditions are

$$w(r,0) = 0, \qquad \frac{\partial w}{\partial\theta}(r,0) = 0$$

$$w(r,\alpha) = 0, \qquad \frac{\partial w}{\partial\theta}(r,\alpha) = 0.$$

For the sake of clarity, we now indicate how basis functions and transcendental equations can be formed for the angular region. These basis functions are used in [1] for finding the deflection of quadrilateral plates and in [2] for finding the frequency of vibration and mode shape of quadrilateral plates.

The method of solution for $\nabla^4 w = \mu w$ can be indicated as follows. Let $w_0(r,\theta)$ be such that $\nabla^4 w_0 = 0$, and $w_1(r,\theta)$ be such that $\nabla^4 w_1 = \mu w_0$. Continuing in this fashion, we define w_{i+1} by $\nabla^4 w_{i+1} = \mu w_i$. Suppose w_i, $i = 1, 2, \ldots$ satisfy the given boundary conditions. Then clearly $w = \sum_{i=0}^{\infty} w_i$ satisfies $\nabla^4 w = \mu w$ and the boundary conditions. We now indicate how this procedure can be applied to problems associated with the angular region.

We have $\nabla^4 w_0 = 0$. The functions w_0 are given in [1,11] for different boundary conditions. Note that on any compact angular region, we can normalize the radius vector such that $|r| < 1$ on the compact region. Choosing

$$w_0(r,\theta) = r^{\lambda+1}(A_0 \cos(\lambda+1)\theta + B_0 \sin(\lambda+1)\theta + C_0 \cos(\lambda-1)\theta + D_0 \sin(\lambda-1)\theta),$$

the clamped boundary conditions imply that the 4×4 determinant

$$\det\begin{pmatrix} 1 & 0 & 1 & 0 \\ \cos(\lambda+1)\alpha & \sin(\lambda+1)\alpha & \cos(\lambda-1)\alpha & \sin(\lambda-1)\alpha \\ 0 & \lambda+1 & 0 & \lambda-1 \\ -(\lambda+1)\sin(\lambda+1)\alpha & (\lambda+1)\cos(\lambda+1)\alpha & -(\lambda-1)\sin(\lambda-1)\alpha & (\lambda-1)\cos(\lambda-1)\alpha \end{pmatrix} = 0.$$

The above equation is equivalent to the transcendental equation

$$\sin^2 \lambda\alpha - \lambda^2 \sin^2 \alpha = 0.$$

Solving the transcendental equation numerically for λ, we get a complex sequence $\{\lambda_j\}_{j=1}^{\infty}$ which we arrange in the order of increasing positive real part.

For each λ, we can determine A_0, B_0, C_0 and D_0 of $w_0(r,\theta)$ upto an arbitrary multiplicative constant (e.g., let $A_0 = 1$) based on the boundary conditions.

Let $\lambda = \lambda_i$ for some i. We need to get w_1 such that

$$\nabla^4 w_1 = \mu w_0 = \mu r^{\lambda+1}(A_0 \cos(\lambda+1)\theta + B_0 \sin(\lambda+1)\theta + C_0 \cos(\lambda-1)\theta + D_0 \sin(\lambda-1)\theta).$$

Hence,

$$\begin{aligned} w_1 = \mu r^{\lambda+5} \bigg[& \frac{A_0 \cos(\lambda+1)\theta + B_0 \sin(\lambda+1)\theta}{[(\lambda+5)^2 - (\lambda+1)^2][(\lambda+3)^2 - (\lambda+1)^2]} \\ & + \frac{C_0 \cos(\lambda-1)\theta + D_0 \sin(\lambda-1)\theta}{[(\lambda+5)^2 - (\lambda+1)^2][(\lambda+3)^2 - (\lambda+1)^2]} \bigg] \\ & + r^{\lambda+5}[A_1 \cos(\lambda+5)\theta + B_1 \sin(\lambda+5)\theta \\ & + C_1 \cos(\lambda+3)\theta + D_1 \sin(\lambda+3)\theta], \end{aligned}$$

where the homogeneous part $r^{\lambda+5}[A_1 \cos(\lambda+5)\theta + B_1 \sin(\lambda+5)\theta + C_1 \cos(\lambda+3)\theta + D_1 \sin(\lambda+3)\theta]$ enables satisfaction of the boundary conditions on $\theta = 0$ and $\theta = \alpha$.

Thus select A_1, B_1, C_1, and D_1 to satisfy the four boundary conditions

$$w_1(r,0) = w_1(r,\alpha) = \frac{\partial w_1}{\partial \theta}(r,0) = \frac{\partial w_1}{\partial \theta}(r,\alpha) = 0.$$

Since λ satisfies the transcendental equation $\sin^2 \lambda\alpha - \lambda^2 \sin^2 \alpha = 0$, to get a nonsingular system for the solution of A_1, B_1, C_1, and D_1, we should have $\sin^2(\lambda+4)\alpha - (\lambda+4)^2 \sin^2 \alpha \neq 0$. This can be easily verified.

Continuing in this manner, we get for $i > 1$,

$$w_i = \tilde{w}_i + r^{\lambda+1+4i}(A_i \cos(\lambda+1+4i)\theta + B_i \sin(\lambda+1+4i)\theta + C_i \cos(\lambda-1+4i)\theta + D_i \sin(\lambda-1+4i)\theta),$$

where $\nabla^4 \tilde{w}_i = \mu w_{i-1}$, and A_i, B_i, C_i, and D_i are selected such that w_i satisfies the boundary conditions along the edges.

3. Differential Equations

Let α be the angle of the angular region and ν be the Poisson's ratio. The transcendental equations for various boundary conditions are given by the following table [1,11].

TABLE 1: Transcendental Equations

Case	Boundary Condition $\theta = 0$	$\theta = \alpha$	Transcendental Equation
1	Clamped	Clamped	$\sin^2 \lambda\alpha = \lambda^2 \sin^2 \alpha$
2	Free	Free	$(3+\nu)^2 \sin^2 \lambda\alpha = (1-\nu)^2\lambda^2 \sin^2 \alpha$
3	Clamped	Free	$(3+\nu)(1-\nu) \sin^2 \lambda\alpha = 4 - (1-\nu)^2\lambda^2 \sin^2 \alpha$
4	Clamped	SS	$\sin 2\lambda\alpha = \lambda \sin 2\alpha$
5	SS	Free	$(3+\nu) \sin 2\lambda\alpha = -(1-\nu)\lambda \sin 2\alpha$
6	SS	SS	$\sin^2 \lambda\alpha = \sin^2 \alpha$

In the table above, SS means simply supported. In cases 1, 2 and 6, the transcendental equations may be further broken down into those for symmetrical and anti-symmetrical modes.

In cases 1,2, and 6, the transcendental equation can be written as

$$a^2 \sin^2 \lambda\alpha = (b\lambda + c)^2 \sin^2 \alpha$$

for suitable constants a, b and c. The transcendental equation for the symmetrical mode is

$$a \sin \lambda\alpha + (b\lambda + c) \sin \alpha = 0.$$

Taking differentials on both sides, we get

$$[\lambda\, d\alpha + \alpha\, d\lambda] a \cos \lambda\alpha + b\, d\lambda \sin \alpha + (b\lambda + c) \cos \alpha\, d\alpha = 0.$$

This results in

$$\frac{d\lambda}{d\alpha} = \frac{-[a\lambda\cos\lambda\alpha + (b\lambda + c)\cos\alpha]}{a\alpha\cos\lambda\alpha + b\sin\alpha}.$$

In the anti-symmetrical case, we get

$$\frac{d\lambda}{d\alpha} = \frac{-[a\lambda\cos\lambda\alpha - (b\lambda + c)\cos\alpha]}{a\alpha\cos\lambda\alpha - b\sin\alpha}.$$

In cases 4 and 5, the transcendental equation can be put in the form

$$a\sin 2\lambda\alpha = b\lambda\sin 2\alpha$$

for suitable a and b. The differential equation can be derived as

$$\frac{d\lambda}{d\alpha} = \frac{2(b\lambda\cos 2\alpha - a\lambda\cos 2\lambda\alpha)}{2a\alpha\cos 2\lambda\alpha - b\sin 2\alpha}.$$

For case 3, the transcendental equation is

$$a\sin^2\lambda\alpha = 4 - b\lambda^2\sin^2\alpha,$$

where

$$a = (3+\nu)(1-\nu),$$
$$b = (1-\nu)^2.$$

The relevant differential equation is

$$\frac{d\lambda}{d\alpha} = \frac{-[b\lambda^2\sin 2\alpha + a\lambda\sin 2\lambda\alpha]}{a\alpha\sin 2\lambda\alpha + 2b\lambda\sin^2\alpha}.$$

The parameter $\mu = \lambda\alpha$ tends to vary more slowly than λ as α is varied from 0 to 2π. Thus, from a computational point of view, it might be beneficial to formulate the differential equations in terms of μ. For the symmetrical mode of cases 1,2, and 6, the differential equation is

$$\frac{d\mu}{d\alpha} = \frac{-(b\mu + c\alpha)\cos\alpha + \frac{b\mu}{\alpha}\sin\alpha}{a\alpha\cos\mu + b\sin\alpha}.$$

For the anti-symmetrical mode of cases 1,2, and 6, the differential equation is

$$\frac{d\mu}{d\alpha} = \frac{(b\mu + c\alpha)\cos\alpha - \frac{b\mu}{\alpha}\sin\alpha}{a\alpha\cos\mu - b\sin\alpha}.$$

The differential equation for cases 4 and 5 is

$$\frac{d\mu}{d\alpha} = \frac{b(2\mu\cos 2\alpha - \frac{\mu}{\alpha}\sin 2\alpha)}{2a\alpha\cos 2\mu - b\sin 2\alpha}.$$

For case 3, the differential equation is

$$\frac{d\mu}{d\alpha} = \frac{b\left(\frac{\mu}{\alpha}\right)^2[-\alpha\sin 2\alpha + 2\sin^2\alpha]}{a\alpha\sin 2\mu + 2b\left(\frac{\mu}{\alpha}\right)\sin^2\alpha}.$$

It can be observed from the numerators of the right sides of the differential equations for μ that as α tends to zero, the numerators in all cases tend to zero also. Thus, for moderate α, the parameter μ is relatively constant.

4. Advantages of Forward Integration

One of the advantages of the proposed method is that if the roots are found at an arbitrary angle, the roots at any other angle in the range of the initial and final angles can also be tabulated with significantly reduced computing effort. Only forward integration is needed to compute the corresponding roots at the intermediate angle. However, as can be observed from the examples in Section 5, the forward integration may not yield all the roots and in fact may occasionally result in spurious roots. We observed that almost all of these missing roots are real and it is not difficult to compute these using any of the standard methods. Some extra effort is needed to compute the missing complex roots and these can be computed using a search procedure. In spite of these obstacles, there is a significant reduction in the computing effort by the use of the forward integration technique since the usual search procedure for all the complex roots is more time consuming and involves an intelligent choice for the initial approximation for the roots.

For moderate α, since the parameter $\mu = \lambda\alpha$ is relatively independent of α, the time taken for the initial approximation of the roots can be reduced. However, when we consider the whole range of $[0, 2\pi]$, the value of μ varies significantly. We present several examples in Section 5 illustrating the variation of μ with α. Our method is especially useful in cases where there is significant variation in μ with α.. It is easy to plot the various values of μ as α varies.

Almost all the complex roots and most of the real roots are obtained by this procedure. However, some of the roots need to be found. It is a lot simpler to find the real roots. One of the observations from Ref. 11 is that at least for $0 < \alpha < \pi$, the real roots are smaller than the least magnitude of the real part of the complex roots for almost all types of boundary conditions. The only exception to this is the clamped-free case. We will present examples involving clamped, clamped-free and simply supported-free boundary conditions in Section 5 to illustrate the usefulness of our approach. It happens on occasion that the forward integration generates spurious roots. Thus it is beneficial to check the validity of the roots obtained by the forward integration procedure.

Another significant advantage with the approach of the differential equations is that the initial value problem can be integrated over the range of α from ϵ to 2π, where ϵ approaches 0. The differential equations become singular at $\alpha = 0$. The roots as α approaches 2π are useful for the solution of the crack problem.

5. Examples

We solve three examples in this section to demonstrate the usefulness of the approach in the case of the three different types of differential equations considered in Section 3.

Example 1: Let us take the transcendental equation for the symmetrical roots given by

$$\sin \lambda\alpha + \lambda \sin \alpha = 0$$

for an angular region with clamped boundary conditions. Note that $\mu = \lambda\alpha$. For $\alpha = 15°$, the first sixteen roots of the transcendental equation in the order of increasing positive real part and the corresponding values of $\frac{\mu}{\pi}$ are evaluated and are given in Table 2.

Let $\tilde{\mu} = \mu/\pi$. To get the value of $\frac{\mu}{\pi}$ at $\alpha = 60°$, the differential equation

$$\frac{d\tilde{\mu}}{d\alpha} = \frac{-\tilde{\mu}\cos\alpha + \frac{\tilde{\mu}}{\alpha}\sin\alpha}{\alpha\cos\pi\tilde{\mu} + \sin\alpha}$$

was solved with the initial condition $\tilde{\mu}(\pi/12)$ given by the values at $\alpha =$ $15°$ given above. We used the fourth order Runge-Kutta method given in the MATLAB package using a tolerance of $1e - 10$. The values of $\tilde{\mu}/\pi$ at $\alpha = 60°, 175°, 225°$ and $355°$ obtained by the forward integration method are listed in Table 3. The real roots that were not generated by the forward integration are listed in Table 4.

TABLE 2: Values of λ and μ/π at $\alpha = 15°$ for the Clamped case	
λ	μ/π
$16.099036079369 + 8.549629866614\,i$	$1.341586339947 + 0.712469155551\,i$
$40.922739395410 + 11.808479416584\,i$	$3.410228282951 + 0.984039951382\,i$
$65.217933871100 + 13.519976459038\,i$	$5.434827822592 + 1.126664704920\,i$
$89.376965549419 + 14.695568014823\,i$	$7.448080462452 + 1.224630667902\,i$
$113.478110132424 + 15.592891970920\,i$	$9.456509177702 + 1.299407664243\,i$
$137.548757675389 + 16.318980707314\,i$	$11.462396472949 + 1.359915058943\,i$
$161.601199318519 + 16.928883919527\,i$	$13.466766609877 + 1.410740326627\,i$
$185.641832424667 + 17.454722092831\,i$	$15.470152702056 + 1.454560174403\,i$
$209.674337397439 + 17.916891466163\,i$	$17.472861449787 + 1.493074288847\,i$
$233.700991180886 + 18.329159178917\,i$	$19.475082598407 + 1.527429931576\,i$
$257.723282631706 + 18.701258444435\,i$	$21.476940219309 + 1.558438203703\,i$
$281.742228750921 + 19.040325717266\,i$	$23.478519062577 + 1.586693809772\,i$
$305.758549248677 + 19.351749656939\,i$	$25.479879104056 + 1.612645804745\,i$
$329.772768581551 + 19.639699129280\,i$	$27.481064048463 + 1.636641594107\,i$
$353.785278459033 + 19.907465712142\,i$	$29.482106538253 + 1.658955476012\,i$
$377.796377663910 + 20.157693813903\,i$	$31.483031471993 + 1.679807817825\,i$

The values of $\tilde{\mu}$ in Table 3 were verified by solving the transcendental equation $\sin\lambda\alpha + \lambda\sin\alpha = 0$ for $\alpha = 60°$ and the resulting match was at least to the tenth decimal place.

Example 2: For the clamped-free case, with $\nu = 0.3$, the transcendental equation is

$$2.31\sin^2\lambda\alpha = 4 - 0.09\lambda^2\sin^2\alpha.$$

The values of λ and μ/π for $\alpha = 15°$ are listed in Table 5. The differential equation

$$\frac{d\tilde{\mu}}{d\alpha} = \frac{0.49\pi\left(\frac{\tilde{\mu}}{\alpha}\right)^2\left[-\alpha\sin 2\alpha + 2\sin^2\alpha\right]}{2.31\alpha\sin 2\pi\tilde{\mu} + 0.98\left(\frac{\pi\tilde{\mu}}{\alpha}\right)\sin^2\alpha}$$

was solved with the initial condition $\tilde{\mu}(\pi/12)$ and the resulting values of $\tilde{\mu}$ for $\alpha = 60°$ and $\alpha = 160°$ are tabulated in Table 6. Table 7 lists the additional roots that were missed by the forward integration procedure. There was also a spurious value of $\tilde{\mu} = 1.550263 + 0.000000i$ generated at $\alpha = 160°$ that does not correspond to a root of the relevant transcendental equation.

TABLE 3: Values of μ/π for various angles: Clamped-Clamped (Sym.) Case

$\alpha = 60°$	$\alpha = 175°$	$\alpha = 225°$	$\alpha = 355°$
$1.353109 + 0.650683\,i$	$1.944236 + 0.000000\,i$	$2.449112 + 0.277056\,i$	$2.028574 + 0.000000\,i$
$3.415242 + 0.925988\,i$	$3.886716 + 0.000000\,i$	$4.461036 + 0.504582\,i$	$4.057384 + 0.000000\,i$
$5.438055 + 1.069263\,i$	$5.825112 + 0.000000\,i$	$6.467937 + 0.629091\,i$	$6.086687 + 0.000000\,i$
$7.450461 + 1.167466\,i$	$7.755302 + 0.000000\,i$	$8.472523 + 0.717154\,i$	$8.116778 + 0.000000\,i$
$9.458395 + 1.242357\,i$	$9.666334 + 0.000000\,i$	$10.475829 + 0.785663\,i$	$10.148027 + 0.000000\,i$
$11.463957 + 1.302929\,i$	$11.491007 + 0.078460\,i$	$12.478341 + 0.841837\,i$	$12.180932 + 0.000000\,i$
$13.468098 + 1.353794\,i$	$13.491502 + 0.202786\,i$	$14.480325 + 0.889485\,i$	$14.216224 + 0.000000\,i$
$15.471314 + 1.397641\,i$	$15.491937 + 0.272454\,i$	$16.481938 + 0.930874\,i$	$16.255098 + 0.000000\,i$
$17.473891 + 1.436174\,i$	$17.492322 + 0.325118\,i$	$18.483278 + 0.967467\,i$	$18.299828 + 0.000000\,i$
$19.476007 + 1.470543\,i$	$19.492667 + 0.368395\,i$	$20.484411 + 1.000267\,i$	$20.356121 + 0.000000\,i$
$21.477779 + 1.501562\,i$	$21.492976 + 0.405490\,i$	$22.485384 + 1.029989\,i$	$22.461252 + 0.000000\,i$
$23.479287 + 1.529826\,i$	$23.493257 + 0.438121\,i$	$24.486230 + 1.057162\,i$	$24.495642 + 0.128506\,i$
$25.480587 + 1.555784\,i$	$25.493512 + 0.467340\,i$	$26.486972 + 1.082192\,i$	$26.495761 + 0.183592\,i$
$27.481720 + 1.579785\,i$	$27.493745 + 0.493849\,i$	$28.487631 + 1.105392\,i$	$28.495873 + 0.224578\,i$
$29.482718 + 1.602103\,i$	$29.493960 + 0.518142\,i$	$30.488219 + 1.127013\,i$	$30.495978 + 0.258322\,i$
$31.483605 + 1.622959\,i$	$31.494158 + 0.540583\,i$	$32.488748 + 1.147255\,i$	$32.496078 + 0.287432\,i$

TABLE 4: Additional values of μ/π for various angles

$\alpha = 60°$	$\alpha = 175°$	$\alpha = 225°$	$\alpha = 355°$
	$1.029416 + 0.000000\,i$	$0.841979 + 0.000000\,i$	$0.986124 + 0.000000\,i$
	$3.089322 + 0.000000\,i$		$2.958268 + 0.000000\,i$
	$5.152843 + 0.000000\,i$		$4.930090 + 0.000000\,i$
	$7.224237 + 0.000000\,i$		$6.901350 + 0.000000\,i$
	$9.314538 + 0.000000\,i$		$8.871763 + 0.000000\,i$
			$10.840970 + 0.000000\,i$
			$12.808479 + 0.000000\,i$

TABLE 5: Values of λ and μ/π at $\alpha = 15°$ for the Clamped-Free case	
λ	μ/π
$10.651024802745 + 0.000000000000\,i$	$0.887585400229 + 0.000000000000\,i$
$7.610592426966 + 1.501062004362\,i$	$0.634216035580 + 0.125088500364\,i$
$22.784846703345 + 6.342239495230\,i$	$1.898737225279 + 0.528519957936\,i$
$35.073902145999 + 8.093194426143\,i$	$2.922825178833 + 0.674432868845\,i$
$47.233021440439 + 9.253448430343\,i$	$3.936085120037 + 0.771120702529\,i$
$59.338504245872 + 10.131592314675\,i$	$4.944875353823 + 0.844299359556\,i$
$71.414977259846 + 10.840846669030\,i$	$5.951248104987 + 0.903403889086\,i$
$83.473543259896 + 11.436757789937\,i$	$6.956128604991 + 0.953063149161\,i$
$95.520120582280 + 11.951029480485\,i$	$7.960010048523 + 0.995919123374\,i$
$107.558209383922 + 12.403569139462\,i$	$8.963184115327 + 1.033630761622\,i$
$119.590034126021 + 12.807735494723\,i$	$9.965836177168 + 1.067311291227\,i$
$131.617085967549 + 13.172944819256\,i$	$10.968090497296 + 1.097745401605\,i$
$143.640406736456 + 13.506091735206\,i$	$11.970033894705 + 1.125507644601\,i$
$155.660748892999 + 13.812380105128\,i$	$12.971729074417 + 1.151031675427\,i$
$167.678670940916 + 14.095836412439\,i$	$13.973222578401 + 1.174653034370\,i$
$179.694597021781 + 14.359641355224\,i$	$14.974549751815 + 1.196636779602\,i$
$191.708855583596 + 14.606352014275\,i$	$15.975737965210 + 1.217196001190\,i$

TABLE 6: Values of μ/π for the Clamped-Free Case	
$\alpha = 60°$	$\alpha = 160°$
$1.079931 + 0.000000\,i$	1.744045 (false root)
$0.567564 + 0.190356\,i$	$0.501101 + 0.245940\,i$
$1.904356 + 0.457347\,i$	$2.505965 + 0.216165\,i$
$2.927638 + 0.611914\,i$	$3.509140 + 0.178136\,i$
$3.940002 + 0.711132\,i$	$4.513539 + 0.096257\,i$
$4.948141 + 0.785436\,i$	$5.363897 + 0.000000\,i$
$5.954037 + 0.845148\,i$	$6.251376 + 0.000000\,i$
$6.958558 + 0.895174\,i$	$7.121553 + 0.000000\,i$
$7.962160 + 0.938271\,i$	$7.978357 + 0.159757\,i$
$8.965111 + 0.976149\,i$	$8.980855 + 0.257304\,i$
$9.967581 + 1.009949\,i$	$9.982578 + 0.320489\,i$
$10.969685 + 1.040473\,i$	$10.983871 + 0.369089\,i$
$11.971501 + 1.068304\,i$	$11.984896 + 0.409186\,i$
$12.973088 + 1.093882\,i$	$12.985740 + 0.443597\,i$
$13.974487 + 1.117546\,i$	$13.986454 + 0.473889\,i$
$14.975733 + 1.139565\,i$	$14.987071 + 0.501033\,i$
$15.976849 + 1.160153\,i$	$15.987613 + 0.525682\,i$

TABLE 7: Additional values of μ/π: Clamped-Free case	
$\alpha = 60°$	$\alpha = 160°$
	$5.677769 + 0.000000\,i$ $6.826798 + 0.000000\,i$ $1.503389 + 0.236691\,i$

Example 3: For the simply supported-free case, with $\nu = 0.3$, the transcendental equation is

$$3.3 \sin 2\lambda\alpha = -0.7\lambda \sin 2\alpha.$$

The values of λ and μ/π for $\alpha = 15°$ are listed in Table 8.

TABLE 8: Values of λ and μ/π at $\alpha = 15°$ for the SS-Free case	
λ	μ/π
$7.895591872816 + 0.000000000000\,i$	$0.657965989401 + 0.000000000000\,i$
$9.307012034278 + 0.000000000000\,i$	$0.775584336190 + 0.000000000000\,i$
$20.718871978887 + 2.743152513018\,i$	$1.726572664907 + 0.228596042752\,i$
$32.777489190922 + 3.676260674920\,i$	$2.731457432577 + 0.306355056243\,i$
$44.813676592684 + 4.288658065852\,i$	$3.734473049390 + 0.357388172154\,i$
$56.838604582663 + 4.748467761566\,i$	$4.736550381889 + 0.395705646797\,i$
$68.856982193617 + 5.117556866893\,i$	$5.738081849468 + 0.426463072241\,i$
$80.871175036583 + 5.426171354656\,i$	$6.739264586382 + 0.452180946221\,i$
$92.882514190978 + 5.691485721814\,i$	$7.740209515915 + 0.474290476818\,i$
$104.891811137801 + 5.924226510378\,i$	$8.740984261483 + 0.493685542532\,i$
$116.899591164329 + 6.131555242723\,i$	$9.741632597027 + 0.510962936894\,i$
$128.906210549635 + 6.318496510883\,i$	$10.742184212470 + 0.526541375907\,i$
$140.911920160849 + 6.488713926984\,i$	$11.742660013404 + 0.540726160582\,i$
$152.916902153721 + 6.644962269170\,i$	$12.743075179477 + 0.553746855764\,i$
$164.921292234989 + 6.789365833965\,i$	$13.743441019582 + 0.565780486164\,i$
$176.925193736710 + 6.923597589457\,i$	$14.743766144726 + 0.576966465788\,i$
$188.928686829963 + 7.048998802756\,i$	$15.744057235830 + 0.587416566896\,i$
$200.931834739295 + 7.166661433097\,i$	$16.744319561608 + 0.597221786091\,i$
$212.934688043977 + 7.277486401963\,i$	$17.744557336998 + 0.606457200164\,i$
$224.937287723129 + 7.382225757249\,i$	$18.744773976927 + 0.615185479771\,i$

TABLE 9: Values of μ/π for the SS-Free Case	
$\alpha = 60°$	$\alpha = 160°$
$0.548912 + 0.000000\,i$	$0.488073 + 0.000000\,i$
$0.915801 + 0.000000\,i$	$1.025131 + 0.000000\,i$
$1.780755 + 0.000000\,i$	$2.050935 + 0.000000\,i$
$2.738015 + 0.155043\,i$	$3.078267 + 0.000000\,i$
$3.739534 + 0.215478\,i$	$4.108517 + 0.000000\,i$
$4.740664 + 0.257281\,i$	$5.144743 + 0.000000\,i$
$5.741543 + 0.289754\,i$	$6.199987 + 0.000000\,i$
$6.742251 + 0.316452\,i$	$7.246253 + 0.074584\,i$
$7.742834 + 0.339177\,i$	$8.246425 + 0.113503\,i$
$8.743324 + 0.358984\,i$	$9.246579 + 0.140984\,i$
$9.743743 + 0.376552\,i$	$10.246719 + 0.163023\,i$
$10.744106 + 0.392343\,i$	$11.246845 + 0.181689\,i$
$11.744424 + 0.406688\,i$	$12.246961 + 0.197998\,i$
$12.744705 + 0.419833\,i$	$13.247067 + 0.212541\,i$
$13.744956 + 0.431965\,i$	$14.247165 + 0.225700\,i$
$14.745181 + 0.443229\,i$	$15.247255 + 0.237739\,i$
$15.745384 + 0.453746\,i$	$16.247339 + 0.248847\,i$
$16.745569 + 0.463602\,i$	$17.247417 + 0.259168\,i$
$17.745738 + 0.472882\,i$	$18.247490 + 0.268812\,i$
$18.745892 + 0.481647\,i$	$19.247558 + 0.277869\,i$

TABLE 10: Additional values of μ/π: SS-Free case	
$\alpha = 60°$	$\alpha = 160°$
$5.072741 + 0.000000\,i$	$1.463953 + 0.000000\,i$
	$2.050935 + 0.000000\,i$
	$3.412327 + 0.000000\,i$
	$4.382660 + 0.000000\,i$
	$6.292132 + 0.000000\,i$

The differential equation

$$\frac{d\tilde{\mu}}{d\alpha} = \frac{0.7\left(\frac{\tilde{\mu}}{\alpha}\right)\left[\sin 2\alpha - 2\alpha\cos 2\alpha\right]}{6.6\alpha\cos 2\pi\tilde{\mu} + 0.7\sin 2\alpha}$$

was solved with the initial condition $\tilde{\mu}(\pi/12)$ and the resulting values of $\tilde{\mu}$ for $\alpha = 60°$ and $\alpha = 160°$ are tabulated in Table 9. Table 10 lists the additional roots that were missed by the forward integration procedure.

6. Conclusions

The roots of the transcendental equations can be found for any arbitrary angular region by solving the associated initial value problem. This procedure is less time consuming since the alternate method of solving the original transcendental equation requires a search procedure in the neighborhood of the roots.

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